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Generalized master equation for systems in nonideal cavities with squeezed baths

R.W.F. van der Plank and L.G. Suttorp[a]

Instituut voor Theoretische Fysica, Universiteit van Amsterdam, Valckenierstraat 65, 1018 XE Amsterdam, The Netherlands

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Abstract. The master equation for the density operator of a system in a lossy cavity, which is coupled to a squeezed bath, is generalized so as to include the effects of an enhanced loss through a mirror of finite transmittivity. As compared to the standard master equation, which is valid for a nearly-perfect cavity, the generalized master equation is found to contain additional terms that account for an effective squeezed-light mixing at the nonideal mirror and for the interplay of the photon loss and the interaction within the cavity. As an example, the new master equation is used to study the influence of the enhanced losses on the photon statistics of a localized degenerate parametric oscillator. It is found that considerable changes in the photon distribution can occur as soon as the quality of the mirror becomes less than perfect.

PACS. 42.50.Ct Quantum description of interaction of light and matter; related experiments – 42.50 Md Optical transient phenomena: quantum beats, photon echo, free-induction decay, dephasings and revivals, optical nutation, and self-induced transparency – 42.50.Dv Nonclassical field states; squeezed, antibunched, and sub-Poissonian states; operational definitions of the phase of the field; phase measurements

1 Introduction

The time evolution of damped quantum systems may be described by means of master equations that incorporate the loss effects through specific damping terms. In particular, master equations have been used frequently for the description of quantum optical systems in lossy cavities. The loss mechanism plays an essential role in analyzing the behavior of these systems, since the radiation escaping the cavity (for instance through a partly transparent mirror) carries the information.

For nearly-perfect cavities, with mirrors of vanishingly small transmittivity, the loss effects lead to damping terms of which the form is well understood [1,2]. If the mirror transmittivity becomes finite, the damping terms get a more complicated structure, owing to the interplay of the loss mechanism and the intracavity interaction. The "generalized" master equation that arises by accounting for these enhanced losses has been established recently [3]. In applying this equation to the time evolution of a decaying atom in a cavity we have found that the effects of increased losses can be quite substantial [4]. These findings are in agreement with the predictions that follow from the solution of delay differential equations for the probability amplitudes [5]. It may be remarked here that the effects of enhanced losses in cavities with a thin-slab geometry and with infinite mirrors, as treated in [6], are rather different from those discussed in [4,5] . In fact, in the slab configuration the atom-field system is always in the weak-coupling regime, so that Fermi's golden rule can be used.

The master equation discussed in [1,2] is valid provided that the losses occur by radiation into a pure vacuum, or, in other words, that the system is coupled to a vacuum "bath". The same configuration has been assumed in deriving the generalized master equation in [3]. Quite different loss effects can be expected if the bath is in a different state, for instance in a squeezed vacuum. This has been demonstrated some time ago [7], by deriving the master equation for a system in a nearly-perfect cavity coupled to a squeezed vacuum.

As in the case of a pure vacuum bath, one may ask how the damping terms for a system coupled to a squeezed bath are modified, if enhanced losses are taken into account. It is the purpose of the present paper to answer this question and to establish a generalized master equation for a system in a nonideal cavity with a squeezed bath. To arrive at our goal we will start from a quantum Langevin equation and analyze the statistical properties of the associated stochastic force.

To show the influence of the increased losses in a specific example we will discuss a localized degenerate parametric oscillator coupled to a squeezed vacuum bath. In this model the medium is confined to a thin slab in the cavity. For the limiting case of a nearly-ideal cavity the properties of a degenerate parametric oscillator in a squeezed bath have been investigated some time ago [8]. Applying the generalized squeezed-bath master equation, we will

[a] e-mail: suttorp@phys.uva.nl

determine the change in the photon statistics brought about by an enhanced loss mechanism.

2 Generalized master equation

We consider a one-dimensional cavity with two mirrors at a distance l. The cavity is assumed to be one-sided, that is, one of the mirrors is perfectly reflecting, whereas the other is semitransparent, with (amplitude) reflectivity r. By putting an additional perfectly reflecting mirror outside the cavity, at a distance L of the semitransparent mirror, a closed "universe" is obtained. The distance L is taken to be much bigger than l and will be sent to ∞ at a later stage.

The electromagnetic field in the universe may be quantized in the usual way, by introduction of a basis of so-called "universe modes". The (positive-frequency part of the) electric field operator inside the cavity then gets the form [2,9]

$$\mathcal{E}(\zeta) = \frac{1}{\sqrt{L}} \sum_k M_k \sin(\omega_k \zeta/2) a_k. \tag{1}$$

Here ζ is the (dimensionless) position inside the cavity; it is 0 at the ideal mirror and 1 at the semitransparent mirror. The sum is extended over all universe modes with (dimensionless) frequencies ω_k, which are equidistant with a separation $2\pi l/L$ in units of the inverse cavity round-trip time $c/(2l)$. Furthermore, the a_k denote the annihilation operators associated to each of the universe modes. Finally, the mode amplitudes M_k are defined by

$$M_k^2 = \frac{(\Gamma/2)(1+\Gamma^2/4)^{1/2}}{\Gamma^2/4 + \sin^2(\Delta\omega_k/2)}, \tag{2}$$

with the width

$$\Gamma = (1-|r|)/|r|^{1/2}. \tag{3}$$

The mode functions depend on the frequency through $\Delta\omega_k$. It is the difference of ω_k and a resonance frequency ω_0, which for a purely dielectric mirror satisfies the relation $\tan(\omega_0/2) = |t|/(1+|r|)$. In defining $\mathcal{E}$ we have suppressed trivial prefactors that depend on $\hbar$, c, or that vary slowly with the frequency.

As is clear from the general form of the mode functions, the universe modes can be grouped in "quasimodes" by choosing consecutive central frequencies ω_0 and confining the frequency differences $|\Delta\omega_k|$ to values smaller than π. In the following we shall concentrate on the dynamical behavior of a single quasimode. To indicate this we write a prime at the symbol for the summation over the universe modes. The weighted sum over the annihilation operators occurring in (1) can be used to define a quasimode annihilation operator $b(\zeta)$, by writing

$$\mathcal{E}(\zeta) = \frac{1}{\sqrt{l}} \mathcal{N}(\zeta) b(\zeta). \tag{4}$$

The normalization factor $\mathcal{N}(\zeta)$ is determined by

$$[\mathcal{N}(\zeta)]^2 = \frac{l}{L} {\sum_k}' \; M_k^2 \sin^2(\omega_k \zeta/2), \tag{5}$$

so that $b(\zeta)$ and its hermitian conjugate satisfy the standard commutation relation.

The above description in terms of quasimodes is particularly useful to study systems in cavities with matter concentrated in such a way that effectively the interaction with the electric field is confined to a single position ζ. For such systems the Hamiltonian H for the interaction between matter and field depends only on the quasimode annihilation operator $b(\zeta)$ and the corresponding creation operator $b^\dagger(\zeta)$, at least in the single-quasimode approximation.

To investigate the dynamical behavior of systems that are governed by a single quasimode a reduced formulation in terms of a master equation for the density operator may be employed. In fact, from the master equation the time evolution of the expectation value of any operator depending on $b(\zeta)$ and $b^\dagger(\zeta)$ can be obtained in a straightforward way. In establishing the master equation one has to specify the properties of the "bath", of which the degrees of freedom are associated to (linear combinations of the) universe modes that are independent of the quasimodes. Often either a vacuum bath or a thermal bath at some finite temperature is considered [1,2]. More recently squeezed baths have been investigated [7].

For one-sided cavities with a semitransparent mirror of a vanishingly small transmittivity, a master equation can be derived easily. If the transmittivity of the semitransparent mirror is small but finite, the master equation gets a more complicated structure. For a vacuum bath its form has been derived recently [3]:

$$\frac{\partial}{\partial\tau}\rho(\tau) = -\frac{i}{\hbar}[H,\rho(\tau)] + \Gamma\left(\mathcal{L}+\mathcal{L}^\dagger\right)\rho(\tau), \tag{6}$$

with τ the time in units of the round-trip time. Here $\rho(\tau)$ is the system density operator. The damping terms are determined by the superoperator $\mathcal{L}$ and its conjugate. This operator is the sum of a standard damping operator $\mathcal{L}_s$ defined by

$$\mathcal{L}_s\rho = [b\rho, b^\dagger], \tag{7}$$

and a correction term $\mathcal{L}_c$ which is a result of the interplay of the Hamiltonian evolution and the damping effects. In leading order in the mirror transmittivity it is given by

$$\mathcal{L}_c\rho = -\frac{1}{2}v(\zeta)\left[\frac{i}{\hbar}[H,b]\rho, b^\dagger\right]. \tag{8}$$

The function $v(\zeta) = v'(\zeta) + iv''(\zeta)$ is defined by

$$v'(\zeta) = -\frac{2\cos(\omega_0\zeta)}{1-\cos(\omega_0\zeta)} \times \left[\zeta - \frac{\sin(\pi\zeta)}{\pi}\left\{\zeta\psi[(\zeta+1)/2] - \zeta\psi(\zeta/2) - 1\right\}\right], \tag{9}$$

$$v''(\zeta) = -2\zeta\frac{\sin(\omega_0\zeta)}{1-\cos(\omega_0\zeta)}, \tag{10}$$

with $\psi(x)$ the digamma function. The factor between the (outer) square brackets in (9) can be shown to be positive.

In the following we wish to generalize the above generalized master equation still further by incorporating the effects of a possible squeezing of the bath. Furthermore, we want to analyze in more detail the transient behavior for small values of τ that has been treated somewhat cavalierly in our earlier papers. As we shall see, these transient effects manifest themselves in a time dependence of Γ for small values of τ.

As a preliminary, we shall review briefly the main points of our earlier derivation [3] of the generalized master equation. One starts from the Heisenberg equation for the time-dependent quasimode annihilation operator

$$b(\zeta,\tau) = \sum_k{}' \ \phi_k(\zeta) a_k(\tau), \tag{11}$$

with the coefficients

$$\phi_k(\zeta) = \sqrt{\frac{l}{L}}\,[\mathcal{N}(\zeta)]^{-1}\, M_k \sin(\omega_k\zeta/2). \tag{12}$$

Suppressing the trivial time dependence associated with the central mode frequency ω_0 one finds an equation of Langevin-type:

$$\begin{aligned}\frac{\partial}{\partial\tau} b(\zeta,\tau) &= -\Gamma b(\zeta,\tau) + \frac{i}{\hbar}\,[H(\zeta,\tau), b(\zeta,\tau)] \\ &+ \int_0^\tau d\tau'\ [\Gamma - \Gamma(\zeta,\tau-\tau')]\, F(\zeta,\tau-\tau') \\ &\times \frac{i}{\hbar}\,[H(\zeta,\tau'), b(\zeta,\tau')] + f(\zeta,\tau),\end{aligned} \tag{13}$$

with the kernel function

$$F(\zeta,\tau) = \sum_k{}' \ \phi_k^2(\zeta) e^{-i\Delta\omega_k\tau} \tag{14}$$

and its logarithmic derivative

$$\Gamma(\zeta,\tau) = -\frac{\partial}{\partial\tau}\log F(\zeta,\tau). \tag{15}$$

Clearly, $F(\zeta,\tau)$ is equal to 1 for $\tau = 0$. The stochastic force in the Langevin equation is

$$f(\zeta,\tau) = \sum_k{}' \ \phi_k(\zeta)(\Gamma - i\Delta\omega_k) e^{-i\Delta\omega_k\tau} a_k(0). \tag{16}$$

As has been shown in [3], the kernel function is proportional to $\exp(-\Gamma\tau)$ for $\tau \gg 1$. In other words, its logarithmic derivative reduces then to Γ. Hence, the combination $\Gamma - \Gamma(\zeta,\tau)$ occurring in (13) is of short range, so that the integral may be replaced by its Markovian approximation. Likewise, it follows that the commutator $[f(\zeta,\tau), b^\dagger(\zeta,0)]$ is of short range. In fact, one derives

$$\left[f(\zeta,\tau), b^\dagger(\zeta,0)\right] = [\Gamma - \Gamma(\zeta,\tau)]\, F(\zeta,\tau), \tag{17}$$

so that the stochastic force operator is associated with degrees of freedom that are independent of the system variables at $\tau = 0$, if transient effects are neglected. In [3,4] we assumed that these bath variables are in a vacuum state, so that we could write $f(\zeta,\tau)\rho_{tot}(0) = 0$, with $\rho_{tot}(0)$ the density operator at $\tau = 0$ for the total system (including the bath).

3 Generalized master equation with a squeezed bath

To show how the form of the generalized master equation changes upon modifying the statistical properties of the bath, we start by redefining the stochastic force. As we have seen above, this force is independent of the system variables, if transient effect are neglected. We want to introduce now instead of $f(\zeta,\tau)$ a modified stochastic force $\bar{f}(\zeta,\tau)$, which strictly commutes with $b^\dagger(\zeta,0)$ for all τ. A way to achieve this is by defining

$$\begin{aligned}\bar{f}(\zeta,\tau) &= \sum_k{}' \ \phi_k(\zeta)\,[\Gamma(\zeta,\tau) - i\Delta\omega_k]\, e^{-i\Delta\omega_k\tau} a_k(0) \\ &= f(\zeta,\tau) - [\Gamma - \Gamma(\zeta,\tau)] \sum_k{}' \ \phi_k(\zeta) e^{-i\Delta\omega_k\tau} a_k(0).\end{aligned} \tag{18}$$

Indeed, from the second form given here one easily proves the commutation relation

$$\left[\bar{f}(\zeta,\tau), b^\dagger(\zeta,0)\right] = 0 \tag{19}$$

for all τ. By employing the formal solution of the Heisenberg equations for $a_k(\tau)$ one arrives at an alternative form for the modified stochastic force:

$$\begin{aligned}\bar{f}(\zeta,\tau) &= f(\zeta,\tau) - [\Gamma - \Gamma(\zeta,\tau)]\, b(\zeta,\tau) \\ &+ [\Gamma - \Gamma(\zeta,\tau)] \int_0^\tau d\tau'\, F(\zeta,\tau-\tau') \\ &\times \frac{i}{\hbar}[H(\zeta,\tau'), b(\zeta,\tau')].\end{aligned} \tag{20}$$

Insertion in (13) yields

$$\begin{aligned}\frac{\partial}{\partial\tau} b(\zeta,\tau) &= -\Gamma(\zeta,\tau) b(\zeta,\tau) + \frac{i}{\hbar}[H(\zeta,\tau), b(\zeta,\tau)] \\ &+ \int_0^\tau d\tau'\, [\Gamma(\zeta,\tau) - \Gamma(\zeta,\tau-\tau')]\, F(\zeta,\tau-\tau') \\ &\times \frac{i}{\hbar}[H(\zeta,\tau'), b(\zeta,\tau')] + \bar{f}(\zeta,\tau).\end{aligned} \tag{21}$$

This form of the Langevin equation is to be preferred to the earlier one, as the stochastic force $\bar{f}(\zeta,\tau)$ occurring here is completely independent of the system degrees of freedom associated with $b(\zeta,0)$ and $b^\dagger(\zeta,0)$.

The integral contribution in the Langevin equation (21) contains a kernel that is proportional to the difference of $\Gamma(\zeta,\tau-\tau')$ and $\Gamma(\zeta,\tau)$. Both these functions tend to Γ as their argument increases, so that their difference is of short range in $\tau-\tau'$. Hence, it is a reasonable approximation to replace the integral by its Markovian form, as before, and to write the Langevin equation in the form

$$\begin{aligned}\frac{\partial}{\partial\tau} b(\zeta,\tau) &= -\Gamma(\zeta,\tau) b(\zeta,\tau) + \frac{i}{\hbar}[H(\zeta,\tau), b(\zeta,\tau)] \\ &+ \chi(\zeta,\tau)\frac{i}{\hbar}[H(\zeta,\tau), b(\zeta,\tau)] + \bar{f}(\zeta,\tau).\end{aligned} \tag{22}$$

Here we introduced the function

$$\chi(\zeta,\tau) = \int_0^\tau d\tau'\ [\Gamma(\zeta,\tau) - \Gamma(\zeta,\tau-\tau')]\, F(\zeta,\tau-\tau'). \tag{23}$$

Having rewritten the Langevin equation in such a form that the stochastic force depends exclusively on the bath degrees of freedom, we can turn to a derivation of the master equation for a squeezed bath. From now on we assume that the total density operator at $\tau = 0$ is given by a state with wideband squeezing, as defined in [10]:

$$\rho_{tot}(0) = S^{-1}(z) \mid 0 \rangle\langle 0 \mid S(z). \qquad (24)$$

Here the wideband squeeze operator is

$$S(z) = \exp\left[-\frac{z}{2}\sum_k{}' \; a_k^\dagger(0)a_{\bar{k}}^\dagger(0) + \frac{z^*}{2}\sum_k{}' \; a_k(0)a_{\bar{k}}(0)\right], \qquad (25)$$

with $\bar{k}$ defined by $\Delta\omega_k = -\Delta\omega_{\bar{k}}$.

For a wideband squeezed state it is no longer true that $\bar{f}(\zeta,\tau)\rho_{tot}(0) = 0$, since one has

$$\left[\cosh(|z|)a_k(0) - \frac{z}{|z|}\sinh(|z|)a_{\bar{k}}^\dagger(0)\right]\rho_{tot}(0) = 0. \qquad (26)$$

One might think that a suitable linear combination of $\bar{f}(\zeta,\tau)$ and its hermitian conjugate can be constructed that gives 0 when acting from the left on the wideband squeezed density operator. However, according to (18) the stochastic force is built up from the annihilation operators $a_k(0)$ with coefficients that are asymmetric under the interchange of k and $\bar{k}$. Hence, no such linear combination of $\bar{f}(\zeta,\tau)$ and its hermitian conjugate can be found. In order to proceed we introduce the symmetric and antisymmetric parts of the coefficients $\phi_k(\zeta)$:

$$\phi_k^{(\pm)}(\zeta) = \frac{1}{2}\left[\phi_k(\zeta) \pm \phi_{\bar{k}}(\zeta)\right]. \qquad (27)$$

Correspondingly, we define the "even" and "odd" parts of the stochastic force by writing

$$\bar{f}(\zeta,\tau) = \bar{f}^{(+)}(\zeta,\tau) + \bar{f}^{(-)}(\zeta,\tau), \qquad (28)$$

with

$$\bar{f}^{(\pm)}(\zeta,\tau) = \sum_k{}' \; \left\{\phi_k^{(\pm)}(\zeta)\left[\mathrm{Re}\,\Gamma(\zeta,\tau) - i\Delta\omega_k\right] + i\phi_k^{(\mp)}(\zeta)\mathrm{Im}\,\Gamma(\zeta,\tau)\right\} e^{-i\Delta\omega_k\tau}a_k(0). \qquad (29)$$

In terms of these two parts of the stochastic force and their hermitian conjugates one may construct suitable linear combinations that yield 0 when acting on the density operator. In fact, one proves

$$\left[\cosh(|z|)\bar{f}^{(\pm)}(\zeta,\tau) \mp \frac{z}{|z|}\sinh(|z|)\bar{f}^{(\pm)\dagger}(\zeta,\tau)\right]\rho_{tot}(0) = 0. \qquad (30)$$

As shown in Appendix A, the properties of the two parts of the stochastic force can be employed to derive a differential equation that governs the time evolution of the expectation value of an arbitrary operator built from the system operators $b(\zeta,\tau)$ and $b^\dagger(\zeta,\tau)$. From that result one derives the following master equation

$$\begin{aligned}\frac{\partial}{\partial\tau}\rho(\tau) &+ \frac{i}{\hbar}[H,\rho(\tau)] \\ &= \Gamma(\tau)\left[(1-\mu)\,\mathcal{L}_z\rho(\tau) + \mu\,\mathcal{L}_{-z}\rho(\tau)\right] \\ &- \chi(\tau)\cosh^2(|z|)\left[\frac{i}{\hbar}[H,b]\rho(\tau), b^\dagger\right] \\ &- \chi(\tau)\sinh^2(|z|)\left[b^\dagger, \rho(\tau)\frac{i}{\hbar}[H,b]\right] \\ &+ \xi(\tau)\frac{z}{|z|}\sinh(|z|)\cosh(|z|)\left[b^\dagger, \rho(\tau)\frac{i}{\hbar}[H,b^\dagger]\right] \\ &+ \xi(\tau)\frac{z}{|z|}\sinh(|z|)\cosh(|z|)\left[\frac{i}{\hbar}[H,b^\dagger]\rho(\tau), b^\dagger\right] \\ &+\text{h.c.}\end{aligned} \qquad (31)$$

All explicit dependence on ζ has been suppressed here. The first term at the right-hand side contains a linear combination of two squeezed damping terms, with superoperators $\mathcal{L}_w$ for $w = z$ and $w = -z$. These are defined as

$$\mathcal{L}_w\rho = \cosh^2(|w|)[b\rho, b^\dagger] + \sinh^2(|w|)[b^\dagger, \rho b] - \frac{w}{|w|}\sinh(|w|)\cosh(|w|)\left([b^\dagger, \rho b^\dagger] + [b^\dagger\rho, b^\dagger]\right). \qquad (32)$$

The linear combination of these superoperators is determined by the (real and positive) parameter

$$\mu(\zeta) = F^{(-)}(\zeta,0) = 1 - F^{(+)}(\zeta,0), \qquad (33)$$

with the functions $F^{(\pm)}$ defined by

$$F^{(\pm)}(\zeta,\tau) = \sum_k{}' \; \phi_k(\zeta)\phi_k^{(\pm)}(\zeta)e^{-i\Delta\omega_k\tau}, \qquad (34)$$

so that one has $F^{(+)} + F^{(-)} = F$ according to (14) and (27). Finally, the master equation contains two (complex) coefficient functions that are integrals over $F(\zeta,\tau)$, $F^{(\pm)}(\zeta,\tau)$ and their derivatives. One of them is $\chi(\zeta,\tau)$, which has been defined already in (23), while the other is:

$$\xi(\zeta,\tau) = \int_0^\tau d\tau'\left[\Gamma(\zeta,\tau) + \frac{\partial}{\partial\tau}\right] \times \left[F^{(+)}(\zeta,\tau-\tau') - F^{(-)}(\zeta,\tau-\tau')\right]. \qquad (35)$$

For small Γ simple approximations for the functions $F^{(\pm)}(\zeta,\tau)$ can be found along the lines of [3]. For $\tau = 0$ one finds

$$F^{(\pm)}(\zeta,0) = \frac{1}{2}(1\pm 1) \pm \frac{1}{4}\Gamma v'(\zeta)\frac{1+\cos(\omega_0\zeta)}{\cos(\omega_0\zeta)}, \qquad (36)$$

with the functions $v'(\zeta)$ and $v''(\zeta)$ as defined in (9)–(10). Hence, the parameter μ becomes for small Γ:

$$\mu = -\frac{1}{4}\Gamma v'(\zeta)\frac{1+\cos(\omega_0\zeta)}{\cos(\omega_0\zeta)}\,. \qquad (37)$$

From (9) it is clear that the right-hand side is positive for all ζ and $\omega_0\zeta$. For $\tau \gg 1$ both functions $F^{(\pm)}(\zeta,\tau)$ are proportional to $\exp(-\Gamma\tau)$, with different prefactors:

$$F^{(\pm)}(\zeta,\tau) = e^{-\Gamma\tau}\left\{\frac{1}{2}(1\pm 1)\left[1+\frac{1}{2}\Gamma v'(\zeta)\right]+\frac{i}{4}\Gamma v''(\zeta)\right\}. \quad (38)$$

Correspondingly, for $\tau \gg 1$ the following approximate time-independent expressions are found for the coefficient functions in (31):

$$\Gamma(\tau) = \Gamma, \quad (39)$$

$$\chi(\tau) = \frac{1}{2}\Gamma v(\zeta) \equiv \chi, \quad (40)$$

$$\xi(\tau) = -\frac{1}{2}\Gamma\frac{v'(\zeta)}{\cos(\omega_0\zeta)} \equiv \xi. \quad (41)$$

Clearly, the coefficients satisfy the relation $\xi - \mathrm{Re}(\chi) = 2\mu$. Substituting these approximate expressions in (31) one ends up with a simplified version of the master equation that is valid only for $\tau \gg 1$:

$$\begin{aligned}\frac{\partial}{\partial\tau}\rho(\tau) &+ \frac{i}{\hbar}[H,\rho(\tau)] \\ &= \cosh^2(|z|)\left[\left\{\Gamma b - \chi\frac{i}{\hbar}[H,b]\right\}\rho(\tau), b^\dagger\right] \\ &+ \sinh^2(|z|)\left[b^\dagger, \rho(\tau)\left\{\Gamma b - \chi\frac{i}{\hbar}[H,b]\right\}\right] \\ &- \frac{z}{|z|}\sinh(|z|)\cosh(|z|) \\ &\quad\times\left[b^\dagger,\rho(\tau)\left\{(1-2\mu)\Gamma b^\dagger - \xi\frac{i}{\hbar}[H,b^\dagger]\right\}\right] \\ &- \frac{z}{|z|}\sinh(|z|)\cosh(|z|) \\ &\quad\times\left[\left\{(1-2\mu)\Gamma b^\dagger - \xi\frac{i}{\hbar}[H,b^\dagger]\right\}\rho(\tau), b^\dagger\right] \\ &\quad + \text{h.c.,}\end{aligned} \quad (42)$$

where (32) has been used. It should be noted that terms with $\Gamma\mu$ have been retained here. They are of the same order of magnitude as the terms proportional to χ and ξ, at least if the (dimensionless) coupling constant in H becomes of the order of Γ. This will indeed be the case in the application of the master equation that we will consider later on in this paper.

If the bath is not squeezed, the master equation (31) reduces to a simpler form:

$$\frac{\partial}{\partial\tau}\rho(\tau) + \frac{i}{\hbar}[H,\rho(\tau)] = \left[\left\{\Gamma(\tau)b - \chi(\tau)\frac{i}{\hbar}[H,b]\right\}\rho(\tau), b^\dagger\right] + \text{h.c.} \quad (43)$$

From this equation one recovers the master equation derived previously [3], at least for $\tau \gg 1$. Indeed, substituting (39-41) in (43), or, alternatively, putting $z=0$ in (42), one arrives at (6) with (7)–(8).

The master equation (31) and its approximate form (42) are the main results of this section. They determine the time evolution of the density operator for the quasimodes in the presence of a squeezed bath. Comparing the new equations with the "standard" master equation that is valid for a system in a nearly-perfect cavity one finds several additional terms, which account for the enhanced losses, namely the terms proportional to μ, χ and ξ. The terms proportional to μ lead to a modification of the "pure" losses, *i.e.* the losses that are independent of any interaction within the cavity. As can be seen from (42), the loss terms that are typical for a squeezed bath (*i.e.* the terms containing either two creation or two annihilation operators) are multiplied by a factor $1-2\mu$. In contrast, the remaining "pure" damping terms, with a single creation and annihilation operator, are left invariant. A detailed discussion of these changes in the "pure" loss mechanism will be given in the next section.

The other two parameters, χ and ξ measure the strength of terms in the master equation that are of a different type. A similar term is present also in the generalized master equation (43) for a system in a nonsqueezed bath. All these terms describe the interplay of the two causes of the time evolution: the coherent interaction within the cavity and the photon losses through the mirror. If the mirror transmittivity is small, these two causes contribute additively. Under those circumstances the photons stay within the cavity for many round-trip times, so that the coherent interaction is felt more or less as in a completely closed cavity. However, if the transmittivity increases, the losses start to perturb the coherent interaction, since the photons do not stay long enough in the cavity anymore. As a consequence, the interaction effects can no longer be described by a simple commutator term, as in the left-hand sides of (42) and (43). The precise way in which the losses and the interaction get mixed up is found to depend on the amount of squeezing: for a nonsqueezed bath it is governed by the parameter χ, whereas in a squeezed environment it is determined by ξ as well.

In the absence of an intracavity interaction the master equation (31) (or its simplified form (42)) can easily be written in the general form given by Lindblad [11]. This is no longer possible when interactions are taken into account. This need not come as a surprise, since in deriving the master equation we had to replace the integral in the Langevin equation (21), which depends on the interaction Hamiltonian, by its Markovian form. As argued in the paragraph above (22), this is a reasonable approximation in view of the time dependence of the relevant integral kernels, but the procedure is not exact. Hence, the Markovian form of the Langevin equation, and of the resulting master equation, can only be approximately valid for interacting systems with enhanced losses. A strictly rigorous theory for an interacting cavity system with enhanced dissipation is expected to be non-Markovian. Nevertheless, a Markovian master equation can yield quite precise predictions for the time evolution of an interacting system even if the damping is considerable, as we have shown in a previous paper [4].

4 Empty cavity with a squeezed bath

The generalized squeezed-bath master equation that we have derived above incorporates the effects of a finite transmittivity of one of the cavity mirrors. The consequences of these effects on the statistics of the quasimode excitations turn out to be nontrivial already in the simplest case, namely that of an empty cavity, as we will show in the following.

For an empty cavity, with $H = 0$, the generalized master equation (42) reduces to

$$\frac{\partial}{\partial\tau}\rho(\tau) = \Gamma\left[(1-\mu)\,\mathcal{L}_z\rho(\tau) + \mu\,\mathcal{L}_{-z}\rho(\tau)\right] + \text{h.c.}, \qquad (44)$$

where we have used the same notation as in (31). At the right-hand side a linear combination of two superoperators $\mathcal{L}_z$ appears, with squeezing parameters z and $-z$. The mixing of these damping operators is determined by the parameter μ defined in (33), so that it is a consequence of the mirror nonideality. Indeed, for the standard master equation [7] one has $\mu = 0$, so that no mixing occurs in that case.

The master equation (44) is valid for large τ. In fact, it describes the evolution of the density matrix in a regime that is established after transient effects have damped out. In the following the dimensionless time τ will be counted from the inception of that regime, so that (44) is valid for all $\tau \geq 0$.

The general time-dependent solution of a master equation of the form (44) has been studied before [12,13] by means of phase space methods. Here we will use a different method in order to elucidate the particular role of the mirror nonideality. We consider the expectation value of a product $\Omega_{p,q} \equiv b^{\dagger p} b^q$ of creation and annihilation operators, with nonnegative integers p, q. The master equation (44) leads to the set of coupled differential equations

$$\begin{aligned}
&\frac{\partial}{\partial\tau}\langle\Omega_{p,q}(\tau)\rangle \\
&\quad = -(p+q)\Gamma\langle\Omega_{p,q}(\tau)\rangle + 2pq\Gamma\sinh^2(|z|)\langle\Omega_{p-1,q-1}(\tau)\rangle \\
&\quad + p(p-1)(1-2\mu)\Gamma\frac{z^*}{|z|}\sinh(|z|)\cosh(|z|)\langle\Omega_{p-2,q}(\tau)\rangle \\
&\quad + q(q-1)(1-2\mu)\Gamma\frac{z}{|z|}\sinh(|z|)\cosh(|z|)\langle\Omega_{p,q-2}(\tau)\rangle,
\end{aligned} \qquad (45)$$

from which $\langle\Omega_{p,q}(\tau)\rangle$ can be determined successively for increasing values of $p + q$.

To get more insight in the solutions of (45) we consider the expectation value of a product $\bar{\Omega}_{p,q} \equiv \bar{b}^{\dagger p}\bar{b}^q$ of a different annihilation operator $\bar{b}$ and its corresponding creation operator. This operator is formally defined as

$$\begin{aligned}
\bar{b}(\tau) = e^{-\Gamma\tau}\, b(0) &+ (1-\mu)^{1/2}\left(1-e^{-2\Gamma\tau}\right)^{1/2}\, c \\
&+ \mu^{1/2}\left(1-e^{-2\Gamma\tau}\right)^{1/2}\, d.
\end{aligned} \qquad (46)$$

The operators c and d are annihilation operators of independent (bath) degrees of freedom, so that one has $[c, c^\dagger] = [d, d^\dagger] = 1$ and $[c, d^\dagger] = [c, b^\dagger(0)] = [d, b^\dagger(0)] = 0$. One may check that $\bar{b}(\tau)$ and its hermitian conjugate satisfy the standard commutation relations.

To define the expectation value of $\bar{\Omega}_{p,q}$ we have to specify the statistics of the bath degrees of freedom. We assume that the state of the degree of freedom corresponding to c is a squeezed vacuum with parameter z, while that corresponding to d is determined by a parameter $-z$. Of course, the density operator of the degree of freedom of $b(0)$ is given by the initial condition of the master equation (44). The expectation value specified in this way will be denoted by $\langle\!\langle \ldots \rangle\!\rangle$.

Since the time dependence of $\bar{b}(\tau)$ is known completely, we can evaluate the time derivative of the expectation value of $\bar{\Omega}_{p,q}$. The result is

$$\begin{aligned}
\frac{\partial}{\partial\tau}\langle\!\langle\bar{\Omega}_{p,q}(\tau)\rangle\!\rangle &= -(p+q)\Gamma\langle\!\langle\bar{\Omega}_{p,q}(\tau)\rangle\!\rangle \\
&+ p\Gamma\left[\left(\frac{1-\mu}{1-e^{-2\Gamma\tau}}\right)^{1/2}\langle\!\langle c^\dagger\bar{\Omega}_{p-1,q}(\tau)\rangle\!\rangle\right. \\
&\left.+\left(\frac{\mu}{1-e^{-2\Gamma\tau}}\right)^{1/2}\langle\!\langle d^\dagger\bar{\Omega}_{p-1,q}(\tau)\rangle\!\rangle\right] \\
&+ q\Gamma\left[\left(\frac{1-\mu}{1-e^{-2\Gamma\tau}}\right)^{1/2}\langle\!\langle\bar{\Omega}_{p,q-1}(\tau)c\rangle\!\rangle\right. \\
&\left.+\left(\frac{\mu}{1-e^{-2\Gamma\tau}}\right)^{1/2}\langle\!\langle\bar{\Omega}_{p,q-1}(\tau)d\rangle\!\rangle\right].
\end{aligned} \qquad (47)$$

Since the state associated to c is a squeezed vacuum, we may write

$$\begin{aligned}
\langle\!\langle\bar{\Omega}_{p,q}(\tau)c\rangle\!\rangle &= \frac{z}{|z|}\sinh(|z|) \\
\times&\left[\cosh(|z|)\langle\!\langle\bar{\Omega}_{p,q}(\tau)c^\dagger\rangle\!\rangle - \frac{z^*}{|z|}\sinh(|z|)\langle\!\langle\bar{\Omega}_{p,q}(\tau)c\rangle\!\rangle\right].
\end{aligned} \qquad (48)$$

Bringing both c and $c^\dagger$ to the left of $\bar{\Omega}_{p,q}(\tau)$ by using the commutation relations, and employing once again the squeezing properties of the c-state, we arrive at the relation

$$\begin{aligned}
&\langle\!\langle\bar{\Omega}_{p,q}(\tau)c\rangle\!\rangle \\
&\quad = p\sinh^2(|z|)(1-\mu)^{1/2}\left(1-e^{-2\Gamma\tau}\right)^{1/2}\langle\!\langle\bar{\Omega}_{p-1,q}(\tau)\rangle\!\rangle \\
&\quad + q\frac{z}{|z|}\sinh(|z|)\cosh(|z|) \\
&\qquad \times(1-\mu)^{1/2}\left(1-e^{-2\Gamma\tau}\right)^{1/2}\langle\!\langle\bar{\Omega}_{p,q-1}(\tau)\rangle\!\rangle.
\end{aligned} \qquad (49)$$

Similar relations can be found for products involving d, $c^\dagger$, and $d^\dagger$. Employing these relations to evaluate (47) we arrive at a set of differential equations which have the same form as (45). Since for $\tau = 0$ one has the trivial identity $\langle\!\langle\bar{\Omega}_{p,q}(0)\rangle\!\rangle = \langle\Omega_{p,q}(0)\rangle$ it follows that the solutions of the differential equations are the same for all $\tau \geq 0$:

$$\langle\Omega_{p,q}(\tau)\rangle = \langle\!\langle\bar{\Omega}_{p,q}(\tau)\rangle\!\rangle. \qquad (50)$$

The identity (50) shows the way in which the expectation value of an arbitrary product $\Omega_{p,q}$ of quasimode creation and annihilation operators changes in time. In fact, by substituting (46) at the right-hand side one sees that as time elapses the expectation value is gradually more dominated by the statistics of the bath degrees of freedom, whereas the initial condition of the quasimode tends to be forgotten. In the following we will concentrate on the statistics for infinite τ.

For $\tau \to \infty$ a simple result for the expectation value emerges:

$$\begin{aligned}&\langle \Omega_{p,q}(\tau \to \infty)\rangle \\ &= \langle\!\langle [(1-\mu)^{1/2}c^\dagger + \mu^{1/2}d^\dagger]^p[(1-\mu)^{1/2}c + \mu^{1/2}d]^q\rangle\!\rangle. \quad (51)\end{aligned}$$

Hence, the statistics of the quasimode in a nonideal cavity for infinite τ is the same as that resulting from the mixing of two squeezed vacua, with opposite squeezing parameters. Such a mixing may be accomplished by a semitransparent mirror. In fact, if such a mirror with an (amplitude) reflection coefficient $(1-\mu)^{1/2}$ and a transmission coefficient $-i\mu^{1/2}$ is irradiated from both sides by photons in a squeezed vacuum mode with identical frequency and squeezing parameter, the statistics of the outgoing photons is the same as that of the quasimode discussed here. The statistics of squeezed light impinging on a semitransparent mirror has been analyzed before with the use of generation functions [14]. From that analysis one derives the following distribution p_n of the quasimode excitations for infinite τ:

$$\begin{aligned}p_n = &\frac{1-\sigma^2}{(1-\sigma^2\rho^2)^{1/2}}\left[\frac{\sigma^2(\sigma^2-\rho^2)}{1-\sigma^2\rho^2}\right]^{n/2} \\ &\times P_n\left(\frac{\sigma(1-\rho^2)}{(1-\sigma^2\rho^2)^{1/2}(\sigma^2-\rho^2)^{1/2}}\right), \quad (52)\end{aligned}$$

where the Legendre polynomials and their prefactors at the right-hand side depend on $\rho \equiv 1-2\mu$ and $\sigma \equiv \tanh(|z|)$. This result may also be derived from the solution presented in [13].

The distribution gets a simple form for a nearly-perfect cavity with $\mu \ll 1$. In that case one finds

$$p_n = \frac{(n-1)!!}{n!!}\sigma^n(1-\sigma^2)^{1/2} \quad (53)$$

for even n, and $p_n = 0$ for odd n. As has been discussed before in the context of squeezed light irradiating a semitransparent mirror [14], the sharp distinction between even and odd n is washed out as soon as the mirror ceases to be perfect, that is, as soon as μ differs from 0. In fact, the statistics is quite sensitive to any nonideality of the mirror: already for an (intensity) reflectivity of 0.975, corresponding to $\mu = 0.025$, the distinction between even and odd n, which is characteristic of a squeezed vacuum, has disappeared almost completely.

The phenomenon that we encounter here, namely that dissipation tends to wash out nonclassical features in the photon statistics of squeezed states, has been discussed before [15–17]. A similar effect has been found to occur in photon counting with less than perfect efficiency [15, 16, 18, 19]. As has been shown in [20], imperfect counting can be incorporated in the theory by convolution of the photon distribution with a binomial distribution. If that procedure is applied to (53), one indeed ends up with a distribution of the form (52), albeit with a different value of σ.

The result (53) would have been obtained directly by starting from the standard master equation for a damped empty cavity in a squeezed bath. As we have seen, it loses its validity for a cavity of which one of the mirrors is less than perfect. Stated otherwise, application of the standard master equation to a nonideal cavity with a squeezed bath may easily lead to wrong results, at least for the empty cavity case. A different and less trivial example will be discussed in the following section.

5 Degenerate parametric oscillator with localized interaction in a squeezed bath

In a degenerate parametric oscillator the presence of a nonlinear medium within the cavity leads to a coupling of a pump mode with a field mode at half frequency. If the pump mode is treated classically, the effective Hamiltonian is proportional to the squared amplitude of the field mode at the position of the nonlinear medium.

Some time ago [8] the effects of a squeezed bath on the statistical properties of a degenerate parametric oscillator have been studied in some detail. In that paper the authors consider a cavity with nearly-perfect mirrors, for which a master equation with standard damping terms [7] is adequate. It was found that the intracavity photon statistics in the stationary regime displays a growing even-odd disparity, as the squeezing parameter of the bath increases.

The generalized master equation described above is a suitable tool to investigate whether the results obtained in [8] for an oscillator in a squeezed bath are robust when the cavity ceases to be ideal. In a recent set of papers [21] the degenerate parametric oscillator in a nonideal cavity has been studied by means of the equations of motion for right- and left-travelling waves. However, the effects of a squeezed bath, which is the main focus of interest of the present paper, were not considered in these articles.

To be able to apply our master equation we shall assume that the medium is concentrated in a thin slice at a position ζ in the cavity. This leads to a model with a localized interaction, for which the interaction Hamiltonian depends exclusively on the annihilation and creation operators of a single quasimode associated to ζ. Adopting furthermore the usual rotating-wave approximation we may write the Hamiltonian of the localized degenerate parametric oscillator as

$$H = i\hbar(\kappa b^2 - \kappa^* b^{\dagger 2}), \quad (54)$$

where the optical frequency has been eliminated by performing a suitable transformation. Here κ is a (complex) coupling constant, which is proportional to the field amplitude in the pump mode. We shall confine ourselves to a discussion of the localized degenerate parametric oscillator below threshold.

As we wish to concentrate on the stationary regime, we may start from the generalized master equation (42), which is valid for large τ. Substituting (54) we get

$$\begin{aligned}
&\frac{\partial}{\partial\tau}\rho(\tau)\\
&= \kappa[b^2,\rho(\tau)] + \cosh^2(|z|)\left[\left(\Gamma b + 2\kappa^*\chi b^\dagger\right)\rho(\tau), b^\dagger\right]\\
&+ \sinh^2(|z|)\left[b^\dagger, \rho(\tau)\left(\Gamma b + 2\kappa^*\chi b^\dagger\right)\right]\\
&- \frac{z}{|z|}\sinh(|z|)\cosh(|z|)\left[b^\dagger, \rho(\tau)\left\{(1-2\mu)\Gamma b^\dagger + 2\kappa\xi b\right\}\right]\\
&- \frac{z}{|z|}\sinh(|z|)\cosh(|z|)\left[\left\{(1-2\mu)\Gamma b^\dagger + 2\kappa\xi b\right\}\rho(\tau), b^\dagger\right]\\
&\qquad +\text{h.c.} \qquad (55)
\end{aligned}$$

In view of our findings of the previous section we want to determine the statistics of the quasimode excitations. The method used above to find the statistics, namely solving a hierarchy of coupled differential equations for the moments by comparison with a similar hierarchy for a case treated before, cannot be employed here straightforwardly, as the hierarchy turns out to be somewhat more complicated. Instead, we shall study the density operator by using its Q-function representation $Q(\beta,\beta^*,\tau) = \pi^{-1}\langle\beta|\rho(\tau)|\beta\rangle$, with $|\beta\rangle$ a coherent state associated to b. It satisfies the Fokker-Planck equation:

$$\begin{aligned}
\frac{\partial}{\partial\tau}Q(\beta,\beta^*,\tau) = \Bigg[&c_1\frac{\partial^2}{\partial\beta^2} + c_1^*\frac{\partial^2}{\partial\beta^{*2}} + 2c_2\frac{\partial^2}{\partial\beta\partial\beta^*}\\
&+c_3\left(\frac{\partial}{\partial\beta}\beta + \frac{\partial}{\partial\beta^*}\beta^*\right) + c_4\frac{\partial}{\partial\beta}\beta^*\\
&+c_4^*\frac{\partial}{\partial\beta^*}\beta\Bigg] Q(\beta,\beta^*,\tau). \qquad (56)
\end{aligned}$$

The coefficients are defined as

$$c_1 = \kappa^* + \Gamma\rho\frac{z}{|z|}\sinh(|z|)\cosh(|z|) - 2\kappa^*\chi\sinh^2(|z|), \quad (57)$$

$$c_2 = \Gamma\cosh^2(|z|) - 2\xi\sinh(|z|)\cosh(|z|)\frac{\mathrm{Re}(\kappa z)}{|z|}, \quad (58)$$

$$c_3 = \Gamma, \quad (59)$$

$$c_4 = 2\kappa^*(1+\chi), \quad (60)$$

with $\rho = 1-2\mu$, as before. Clearly, c_1 and c_4 are complex, whereas c_2 and c_3 are real.

We are interested in particular in the stationary solution that is reached as τ tends to ∞. To find it we introduce the characteristic function $\tilde{Q}(\lambda,\lambda^*) = \langle\exp(\lambda b^\dagger) \times\exp(-\lambda^* b)\rangle$, which is proportional to the Fourier transform of the Q-function [22]. Its stationary form is Gaussian: $\tilde{Q}(\lambda,\lambda^*) = \exp(A\lambda^2 + A^*\lambda^{*2} + 2B\lambda\lambda^*)$. The coefficients are found as:

$$A = \frac{(c_1c_4^* - c_1^*c_4)c_4^* + 2(c_1^*c_3 - c_2c_4^*)c_3}{4c_3(c_3^2 - |c_4|^2)}, \quad (61)$$

$$B = \frac{(c_1 - c_4)c_4^* + (c_1^* - c_4^*)c_4 - 2(c_2 - c_3)c_3}{4(c_3^2 - |c_4|^2)}\cdot \quad (62)$$

Table 1. The minimum standard deviation of the field quadrature X_α for a localized parametric oscillator with a squeezed vacuum bath of squeezing parameter $|z| = 2$, at various values of the interaction parameter $s = \kappa z/(\Gamma|z|)$ and of the dissipation parameter Γ. The position of the oscillator is given by $\zeta = 0.5$ and $\omega_0\zeta = \pi/2 \pmod{2\pi}$.

s	$\Gamma = 0.0$	$\Gamma = 0.01$	$\Gamma = 0.1$
0.00	0.0046	0.0263	0.2217
−0.25	0.0031	0.0248	0.2190
−0.49	0.0023	0.0240	0.2151
0.25	0.0092	0.0309	0.2312
0.49	0.2289	0.2627	1.7147
0.25 i	0.9015	0.9028	0.9148
−0.25 i	0.9015	0.9363	1.2553
0.49 i	3.3103	3.2934	3.1442
−0.49 i	3.3103	3.3290	3.4897

As the characteristic function is the generating function of the moments of the creation and annihilation operators, one easily obtains the standard deviation of a field quadrature $X_\alpha = (e^{i\alpha}b + e^{-i\alpha}b^\dagger)/2$, with arbitrary α. Varying α we find the minimum value of the standard deviation as

$$\langle(\Delta X_\alpha)^2\rangle_{min} = -|A| - B + 1/4. \quad (63)$$

In Table 1 we list the minimum standard deviation for the case of a squeezed bath, at several values of the interaction parameter $|s| = |\kappa|/\Gamma$ and the dissipation parameter Γ. For an ideal cavity the minimum standard deviation can be written as:

$$\begin{aligned}
&\langle(\Delta X_\alpha)^2\rangle_{min}\\
&= \frac{1 + \sigma^2 - 4\sigma\,\mathrm{Re}(s) - 2|s(1+\sigma^2) + \sigma(2|s|^2 - 2s^2 - 1)|}{4(1-\sigma^2)(1-4|s|^2)}, \qquad (64)
\end{aligned}$$

with $\sigma = \tanh(|z|)$ and $s = \kappa z/(\Gamma|z|)$. If s is real and negative, the minimum standard deviation decreases for increasing $|s|$. In particular, for the ideal-cavity case $\langle(\Delta X_\alpha)^2\rangle_{min}$ decreases by a factor 2, when $|s|$ goes to its threshold value, as in the case with a nonsqueezed bath [8]. If the cavity becomes nonideal, however, the standard deviation changes much less. More importantly, the range of values of $\langle(\Delta X_\alpha)^2\rangle_{min}$ for the nonideal cavity is quite different from that found in the ideal case. In fact, for $\Gamma = 0.1$ the minimum standard deviation for $s = 0$ is already near 0.25. Clearly, the squeezing due to the bath is spoilt by the mixing effects at the semitransparent mirror, as has been discussed already in the previous section.

If the phase of the (complex) interaction parameter s is different from π, the picture changes. No longer do we see a decrease of $\langle(\Delta X_\alpha)^2\rangle_{min}$ for $|s|$ going to threshold. On the contrary, it increases for all cases that have been tabulated. For real and positive s we still find squeezing for small values of s in the ideal cavity, but it is destroyed rapidly as the cavity becomes nonideal. For purely imaginary s we find rather large values of the minimum standard deviation.

The characteristic function can be used as well to determine the probability distribution of the quasimode excitations. Expanding the (Gaussian) characteristic function in powers of λ and λ^* and using the identity [24]:

$$\sum_{k=0}^{[n/2]} \frac{n!}{(n-2k)!(k!)^2}\left(\frac{z^2}{4}\right)^k = (1-z^2)^{n/2}\, P_n[(1-z^2)^{-1/2}], \tag{65}$$

with P_n the Legendre polynomials, one finds an explicit expression for the expectation value of a product of creation and annihilation operators:

$$\langle b^{\dagger n} b^n\rangle = n!(-2B)^n\left(1-\frac{|A|^2}{B^2}\right)^{n/2} \times P_n\left(\left[1-\frac{|A|^2}{B^2}\right]^{-1/2}\right). \tag{66}$$

From this result one easily derives the probability distribution. First one defines the generating function

$$G(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}\langle b^{\dagger n} b^n\rangle, \tag{67}$$

of which the explicit form is [24]:

$$G(x) = \left[1+4xB+4x^2(B^2-|A|^2)\right]^{-1/2}. \tag{68}$$

Subsequently, one uses the relation

$$G(x-1) = \sum_{n=0}^{\infty} x^n p_n \tag{69}$$

to derive the quasimode probability distribution:

$$p_n = \frac{1}{[(2B-1)^2-4|A|^2]^{1/2}}\left[\frac{4(B^2-|A|^2)}{(2B-1)^2-4|A|^2)}\right]^{n/2} \times P_n\left(\frac{2(B^2-|A|^2)-B}{(B^2-|A|^2)^{1/2}[(2B-1)^2-4|A|^2]^{1/2}}\right). \tag{70}$$

For $\kappa = 0$ one gets from (61)–(62):

$$A = \frac{1}{2}\rho\frac{z^*}{|z|}\sinh(|z|)\cosh(|z|),$$
$$B = -\frac{1}{2}\sinh^2(|z|). \tag{71}$$

Inserting these expressions in (70) one recovers (52).

For $\kappa \neq 0$ the quasimode probability distribution is a function of s, σ, ρ, χ and ξ, which is rather cumbersome to study analytically. The expression simplifies considerably for the ideal case of small cavity losses. Then both Γ and κ are vanishingly small, at finite s, so that one has $\rho = 1$, $\chi = 0$ and $\xi = 0$. The probability distribution is found as

$$p_n = (1-\sigma^2)\left(\frac{1-4|s|^2}{U}\right)^{1/2}\left(\frac{V}{U}\right)^{n/2} P_n\left(\frac{W}{(UV)^{1/2}}\right), \tag{72}$$

with the abbreviations

$$U = -\sigma^4|s|^2 + 2\sigma^3\mathrm{Re}(s) + \sigma^2[4|s|^2 - 2\mathrm{Re}(s^2) - 1] - 2\sigma\mathrm{Re}(s) + 1 - |s|^2, \tag{73}$$

$$V = \sigma^4(1-|s|^2) - 2\sigma^3\mathrm{Re}(s) + \sigma^2[4|s|^2 - 2\mathrm{Re}(s^2) - 1] + 2\sigma\mathrm{Re}(s) - |s|^2, \tag{74}$$

$$W = \sigma^4|s|^2 - 2\sigma^2\mathrm{Re}(s^2) + |s|^2. \tag{75}$$

Normalization of the probabilities implies the relation $U + V - 2W = (1-4|s|^2)(1-\sigma^2)^2$, which is easily checked. Both U and W are positive for $0 < \sigma < 1$ and $|s| < 1/2$. The sign of V depends on the values of s and σ. Clearly, the ideal case discussed here is covered by the standard master equation. Indeed, the result (72) is equivalent to that obtained in [8] by means of the latter equation. For a nonsqueezed bath the photon distribution derived in [25] is recovered.

The change in the quasimode probability distribution for increasing mirror transmittivity can be seen by evaluating (70) numerically. An example is given in Figure 1. Comparison of the distributions for an ideal cavity with $\Gamma \downarrow 0$ and nonideal cavities with $\Gamma = 0.01$ and 0.1, for the same value of s, shows that the influence of the mirror transmittivity can be considerable. As in the previous section it is found that the even-odd disparity in the distribution may disappear nearly completely as a consequence of the leakage from the cavity, even though the squeezing of the bath is rather high.

In conclusion, we have found that, when using a master equation to study quantum optical systems in nonideal cavities, it is essential to start from an equation in which the effects of the enhanced dissipation are incorporated systematically. The "standard" versions of the master equation that have been formulated for systems in nearly-ideal cavities [1,2,7] are not adequate for this purpose. For systems with nonsqueezed baths we have reached this conclusion before [3,4]. In the present paper we have shown that the same holds true for squeezed baths.

Appendix A: Derivation of master equation

To derive the master equation (31) we start from the Langevin equation (22) in Markovian form. It contains the modified stochastic force $\bar{f}(\zeta,\tau)$ for which an expression in terms of $b(\zeta,\tau)$ and $f(\zeta,\tau)$ has been given in (20). From the formal solution of the Heisenberg equations for $a_k(\tau)$

$$a_k(\tau) = e^{-i\Delta\omega_k\tau}a_k(0) + \phi_k(\zeta)\int_0^\tau d\tau'\, e^{-i\Delta\omega_k(\tau-\tau')}\frac{i}{\hbar}[H(\zeta,\tau'),b(\zeta,\tau')] \tag{A.1}$$

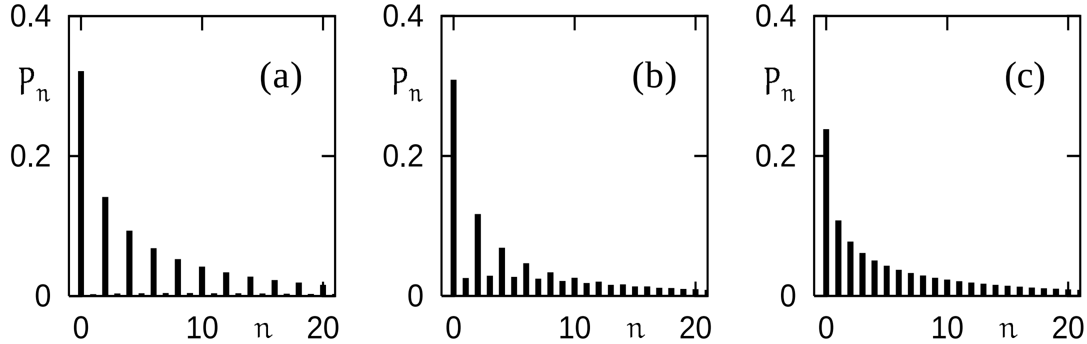

Fig. 1. The probability distributions p_n for quasimode excitations of a localized parametric oscillator in cavities with (a) $\Gamma \downarrow 0$, (b) $\Gamma = 0.01$, and (c) $\Gamma = 0.1$. The squeeze parameter of the bath and the interaction parameter are $z = 2$ and $s = 0.25$, respectively. The position of the oscillator is given by $\zeta = 0.5$ and $\omega_0\zeta = \pi/2 \pmod{2\pi}$.

one may derive

$$f(\zeta,\tau) = {\sum_k}' \ \phi_k(\zeta)(\Gamma - i\Delta\omega_k)a_k(\tau) - \int_0^\tau d\tau' \ [\Gamma - \Gamma(\zeta,\tau-\tau')]\, F(\zeta,\tau-\tau') \times \frac{i}{\hbar}\,[H(\zeta,\tau'),b(\zeta,\tau')]\,. \quad (A.2)$$

As a consequence the modified stochastic force can be rewritten as

$$\bar{f}(\zeta,\tau) = {\sum_k}' \ \phi_k(\zeta)[\Gamma(\zeta,\tau) - i\Delta\omega_k]a_k(\tau) - \int_0^\tau d\tau' \,[\Gamma(\zeta,\tau) - \Gamma(\zeta,\tau-\tau')]\, F(\zeta,\tau-\tau') \times\frac{i}{\hbar}[H(\zeta,\tau'),b(\zeta,\tau')]. \quad (A.3)$$

Applying the Markov approximation in the integral, as before, we arrive at

$$\bar{f}(\zeta,\tau) = {\sum_k}' \ \phi_k(\zeta)[\Gamma(\zeta,\tau) - i\Delta\omega_k]a_k(\tau) - \chi(\zeta,\tau)\frac{i}{\hbar}[H(\zeta,\tau),b(\zeta,\tau)], \quad (A.4)$$

with $\chi(\zeta,\tau)$ as defined in (23). From this expression one derives the commutation relation

$$\left[b(\zeta,\tau), \bar{f}(\zeta,\tau) + \chi(\zeta,\tau)\frac{i}{\hbar}[H(\zeta,\tau),b(\zeta,\tau)]\right] = 0. \quad (A.5)$$

Expressions analogous to (A.4) can be established for the "even" and "odd" parts of the stochastic force:

$$\bar{f}^{(\pm)}(\zeta,\tau) = {\sum_k}' \ \left\{\phi_k^{(\pm)}(\zeta)[\mathrm{Re}\,\Gamma(\zeta,\tau) - i\Delta\omega_k] + i\phi_k^{(\mp)}(\zeta)\mathrm{Im}\,\Gamma(\zeta,\tau)\right\} a_k(\tau) - \left[\chi^{(\pm)\prime}(\zeta,\tau) + i\chi^{(\mp)\prime\prime}(\zeta,\tau)\right] \frac{i}{\hbar}[H(\zeta,\tau),b(\zeta,\tau)], \quad (A.6)$$

with the abbreviations

$$\chi^{(\pm)\prime}(\zeta,\tau) = \int_0^\tau d\tau' \left\{\mathrm{Re}[\Gamma(\zeta,\tau)] + \frac{\partial}{\partial\tau}\right\} F^{(\pm)}(\zeta,\tau-\tau'), \quad (A.7)$$

$$\chi^{(\pm)\prime\prime}(\zeta,\tau) = \int_0^\tau d\tau' \,\mathrm{Im}[\Gamma(\zeta,\tau)]F^{(\pm)}(\zeta,\tau-\tau'). \quad (A.8)$$

Let us consider now the time derivative of the expectation value of an arbitrary system operator $\Omega(\zeta,\tau)$ depending on $b(\zeta,\tau)$ and $b^\dagger(\zeta,\tau)$. From now on we will suppress again the variable ζ. Employing the Langevin equation (22) and the commutation relation (A.5) we arrive at

$$\frac{\partial}{\partial\tau}\langle\Omega(\tau)\rangle = \frac{i}{\hbar}\langle[H(\tau),\Omega(\tau)]\rangle - \Gamma(\tau)\langle[\Omega(\tau),b^\dagger(\tau)]b(\tau)\rangle - \Gamma^*(\tau)\langle b^\dagger(\tau)[b(\tau),\Omega(\tau)]\rangle + \chi(\tau)\frac{i}{\hbar}\langle[\Omega(\tau),b^\dagger(\tau)][H(\tau),b(\tau)]\rangle + \chi^*(\tau)\frac{i}{\hbar}\langle[H(\tau),b^\dagger(\tau)][b(\tau),\Omega(\tau)]\rangle + \langle[\Omega(\tau),b^\dagger(\tau)]\bar{f}(\tau)\rangle + \langle\bar{f}^\dagger(\tau)[b(\tau),\Omega(\tau)]\rangle. \quad (A.9)$$

To evaluate the terms with the stochastic force we write

$$\bar{f}(\tau) = \cosh(|z|)\bar{f}'(\tau) + \frac{z}{|z|}\sinh(|z|)\bar{f}''(\tau), \quad (A.10)$$

with

$$\bar{f}'(\tau) = \cosh(|z|)\left[\bar{f}^{(+)}(\tau) + \bar{f}^{(-)}(\tau)\right] - \frac{z}{|z|}\sinh(|z|)\left[\bar{f}^{(+)\dagger}(\tau) - \bar{f}^{(-)\dagger}(\tau)\right], \quad (A.11)$$

$$\bar{f}''(\tau) = \cosh(|z|)\left[\bar{f}^{(+)\dagger}(\tau) - \bar{f}^{(-)\dagger}(\tau)\right] - \frac{z^*}{|z|}\sinh(|z|)\left[\bar{f}^{(+)}(\tau) + \bar{f}^{(-)}(\tau)\right]. \quad (A.12)$$

As a consequence of (30) we have $\bar{f}'(\tau)\rho_{tot}(0) = 0$ and $\bar{f}''^\dagger(\tau)\rho_{tot}(0) = 0$, so that the terms with $\bar{f}(\tau)$ and its hermitian conjugate in (A.9) can be written as

$$\frac{z}{|z|}\sinh(|z|)\left\langle\left[[\Omega(\tau),b^\dagger(\tau)],\bar{f}''(\tau)\right]\right\rangle + \frac{z^*}{|z|}\sinh(|z|)\left\langle\left[\bar{f}''^\dagger(\tau),[b(\tau),\Omega(\tau)]\right]\right\rangle. \quad (A.13)$$

The commutators are found by starting from (A.6) and using the standard commutation relations for the creation and annihilation operators. One finds then

$$\langle[\Omega(\tau),\bar{f}''(\tau)]\rangle = -\cosh(|z|)\xi(\tau)\frac{i}{\hbar}\langle[\Omega(\tau),[H(\tau),b^\dagger(\tau)]]\rangle + \frac{z^*}{|z|}\sinh(|z|)\chi(\tau)\frac{i}{\hbar}\langle[\Omega(\tau),[H(\tau),b(\tau)]]\rangle + \cosh(|z|)\Gamma(\tau)(1-2\mu)\langle[\Omega(\tau),b^\dagger(\tau)]\rangle - \frac{z^*}{|z|}\sinh(|z|)[\Gamma(\tau) - i\sum_k{}' \phi_k^2 \Delta\omega_k]\langle[\Omega(\tau),b(\tau)]\rangle. \quad (A.14)$$

Replacing $\Omega(\tau)$ by $[\Omega(\tau),b^\dagger(\tau)]$, adding a similar expression with $\bar{f}''^\dagger(\tau)$, and inserting the result in (A.9), we finally arrive at an identity for the time derivative of $\langle\Omega(\tau)\rangle$. Since $\Omega(\tau)$ is an arbitrary system operator, the identity can be rephrased as a time evolution equation for the density operator. Its form is found to be that of (31), which is thus proved.

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